**Homework 01: Fundamental Concepts**

**PHYS550 – Quantum Mechanics I**

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***Additional Texts Referenced: Introduction to Quantum Mechanics, Griffiths and Schroeter***

**Problem 1.2**

*Prove [AB,CD] = -AC{D,B} + A{C,B}D – C{D,A}B + {C,A}DB.*

Noting that [X,Y] = XY – YX, {X,Y} = XY + YX, and [AB,C] = A[B,C] + [A,C]B, we need to show the equality to be true. Note that A, B, C, D are operators so multiplication is not commutative, but addition is.

[AB,CD] = -AC{D,B} + A{C,B}D – C{D,A}B + {C,A}DB

=> ABCD – CDAB = -AC(DB+BD) +A(CB+BC)D – C(DA+AD)B + (CA+AC)DB

=> ABCD – CDAB = - ACDB – ACBD +ACBD + ABCD – CDAB - CADB + CADB + ACDB

=> 0 = - ACDB – ACBD +ACBD - CADB + CADB + ACDB

=> 0 = AC(-DB–BD+BD+DB) + CA(-DB + DB)

=> 0 = 0

Thus, we have proven the original statement as 0=0 and every step in the process is reversible.

**Problem 1.6**

*Using the rules of bra-ket algebra, prove or evaluate the following:*

*a) tr(XY) = tr(YX), where X and Y are operators.*

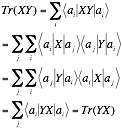
The transpose is defined as the sum of the diagonal terms of a matrix. We can represent this with bra-ket algebra as:



To fully prove this, we need the definition of the “complicated one.”



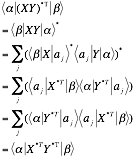
Which allows us to insert the complicated one anywhere we want. So…



Which proves what we set out to prove. This is only possible because we know that things of the form “bra-operator-ket” are complex numbers and thus multiplication is commutative between them.

*b) (XY)†=Y†X†, where X and Y are operators.*

Keep in mind that the dagger means “Hermitian adjoint,” which in practice just means the matrix that represents the operator is flipped and gets its conjugate taken. Due to the difficulty in typing the dagger symbol, it will most often just be represented by the equivalent: \*T. It’s often best to think of operators acting on certain vectors, so for this problem we will place generic vectors on either side like so:



And thus (XY)†=Y†X† is proven. Note that this relies on the complicated one provided in part a), and the definition of the transpose.

*c) exp[if(A)]=? In ket-bra form, where A is a Hermitian operator whose egienvalues are known.*

NOTE: I was at first very confused about this: what kind of function was f(A)? We know A is Hermitian, but the function itself? A quick Internet search provided the realization that, yes, it really is just a completely generic function. As for how to find the answer, all I saw was that it needed a ket added to it, which was a reasonable assumption given the nature of the problem.

SECOND NOTE: So I’m fairly certain this answer is wrong, so the searches I performed above clearly didn’t help. What follows is my best guess as to what the problem even wants.

When given a ket to act on, we obtain:



Since we can treat the entire function as some kind of operator, we bring the ket in:



And, so far as we can tell, we are stuck here. While hermitian operators do have a property that allows for ** (From Griffiths) we do not know the form of the function f(). We know it can take an argument of a ket, but for all we know f could be represented by either a “bra” or a full operator, which would change the result significantly. So we go ahead and make an additional assumption: since we are looking for a specific value in the answer, we can deduce that f() *should* output a complex number of some kind in order for the exponential to make sense. (What exactly would raising something to the power of a matrix even do? Alter every single element in the matrix?)

Once we make this assumption, we may now write:



Multiplying a complex number by i has an interesting property: a+bi becomes ai-b, which is just another complex number. Since the function f is perfectly generic we can pull the i into it and just end up with the result:



Which is just ec where c is some complex number—again, c could be literally anything due to the nature of f. This includes c being entirely real, or entirely complex. (The multiplication of “i” does not change this fact. f could be a function as brutal as f(A) = i or f(A) = 1 that doesn’t even care what A is). It seems as if there is **no meaningful observation that can be made about this equation.**

A bunch of other methods turned up nothing particularly helpful either.

So far as I can tell, the best we can say is that “The hermitian operator will translate a given ket into another ket, at which point the function f(x) takes over.”

Which can effectively be stated as:

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Where the subscript just indicates that every component of the ket has been multiplied by the eigenvalue associated with that component.

Very clearly I’m missing something obvious here, but I have no idea what.

One of the other students suggested expanding the exponential as an infinite sum, but that did not simplify the equation at all, it made it quite a bit more complicated and still suffered from the same problem of knowing nothing about f().

*d)  where *

This can easily be re-written as:

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And then:

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Which is just the inner product of our two input parameters.

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**Problem 1.12**

*The Hamiltonian operator for a two-state system is given by*

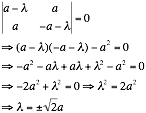
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*where a is a number with the dimension of energy. Find the energy eigenvalues and the corresponding energy eigenkets (as linear combinations of  and ).*

We note that the arrangements of bras and kets above is equivalent to that of a matrix, specifically one with the following arrangement:

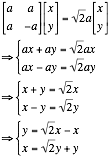
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Then we just go through the standard methods to find the eigenvalues and eigenkets (or eigenvectors, since we’re using matrix notation now.) First of all, subtract λ from the diagonal and solve for zero determinant:

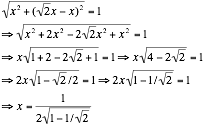
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| Eigenvalues: +a√2, -a√2 |

Now that we have obtained the eigenvalues, we are prepared to find the corresponding eigenket (eigenvector) for each value through the wonders of matrix multiplication.

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With this relation we can now normalize for one variable: let’s make it x.

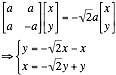
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And then we can also solve for y.

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Which isn’t pretty, but it is x and y for our first eigenvector.

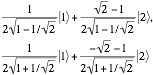
For the second eigenvector, only a single sign is changed.

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Which will just flip the minus sign in the final x component, so we can simply adjust our eigenvectors to match. The y coordinate is similarly easy to find.

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Which are the eigenkets for our previously established eigenvalues. A quick check confirms that they are in fact normalized. If we write them as linear combinations, we get:

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